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A SIMPLE EXISTENCE-PROOF FOR LOGARITHMS.

By JOHN WESLEY YOUNG, Instructor in Mathematics, Northwestern University.

Professor Osgood in an introduction to a recent paper by Mr. Bradshaw* says: "The student of mathematics and physics meets logarithms for the first time at an early stage. He is told that 'the logarithm of a number is the exponent of the power to which a certain number, taken as the base, must be raised in order to equal the given number.' The definition is purely formal. Probably the beginner has never seen a proof for the existence even of the fifth root of 2, ..., and if he has, it is not likely that it has meant anything to him. He tacitly assumes that every number has a positive qth root, q being any positive integer. It is then an easy step to any rational power, and the irrational powers are thought of as limiting cases, the principle being that, whenever one wants a limit in mathematics, the limit exists.

"Now all of these assumptions have been justified by rigorous ϵ-proofs in well known treatises on modern analysis. But the general student of mathematics and physics does not read these proofs, for they are uninteresting to him; and thus the great majority of students of the Calculus never see a proof that there is such a thing as a logarithm."

Mr. Bradshaw, in the paper cited, gives a proof by means of elementary calculus. In the following we give a proof which, while rigorous, makes use of only elementary algebra and which is much shorter than the proofs usually given.†

But first let us state precisely what is required, from an arithmetic point of view, to prove, the existence of such numbers as rational and irrational powers, logarithms, etc. Cantor's definition of an irrational number is as follows.‡ Suppose we have a series of rational numbers

$$\phi_0,\ \phi_1,\ \phi_2,\ \cdots\cdots,\ \phi_n,\ \cdots\cdots \qquad (1)$$

with the property that for every number $\epsilon > 0$ we may find a number μ such that the relation

$$|\ \phi_{n+r} - \phi_n\ | < \epsilon$$

is satisfied, provided only that n be greater than μ, r being any positive integer; then if $\underset{n \doteq \infty}{\text{Lim}} [\phi_n]$ is not a rational number, the series (1) *defines* an irrational number equal to $\underset{n \doteq \infty}{\text{Lim}} [\phi_n]$. In order to prove that a certain irrational number a *exists* it is necessary to establish a series (1) such that $|\ a - \phi_n\ | < \epsilon$, provided only that $n > \mu$.

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*J. W. Bradshaw, The Logarithm as a Direct Function; *Annals of Mathematics* (January, 1903), 2nd series, v. 4, p. 51.

†Stolz, in his *Allgemeine Arithmetik*, devotes 24 pages to the discussion of powers, roots and logarithms; cf. Vol. I, Chap. 8, pp. 125-148.

‡Stolz, *Allgemeine Arithmetik*, I, p. 105.

We now turn to the problem of proving the existence of a logarithm of any positive number. Of irrational operations we shall assume only the possibility of extracting the square root of any positive number; that such a root exists may be seen geometrically or (better) the ordinary ϵ-proof may be supplied, which in this case is very simple.* Consider two sequences of numbers G_0 and A_0, the former in geometric, the latter in arithmetic progression:

$$(G_0) \quad \ldots\ldots, a^{-m}, \ldots\ldots, a^{-2}, a^{-1}, 1, a, a^2, \ldots\ldots, a^n, \ldots\ldots$$

$$(A_1) \quad \ldots\ldots, -mb, \ldots\ldots, -2b, -b, 0, b, 2b, \ldots\ldots, nb, \ldots\ldots$$

where a, b are any positive numbers ($a \neq 1$). Establish a one-to-one correspondence between the terms of the sequences, such that 1 of G_0 corresponds to 0 of A_0, a of G_0 to b of A_0, and in general a^k of G_0 to kb of A_0, for all integral, positive or negative, values of k. Then if p, q be any two numbers of G_0, and if $\phi(p)$, $\phi(q)$ denote the corresponding numbers of A_0, we have

$$\phi(pq) = \phi(p) + \phi(q). \tag{2}$$

It is our object to prove the existence of numbers $\phi(x)$, which have the property (2), for all real positive values of x. To do this insert a geometric mean between some pair of consecutive numbers of G_0, and corresponding thereto insert an arithmetic mean between the corresponding pair of A_0, and suppose this done for every pair of consecutive numbers of the two sequences. We thus obtain two new corresponding sequences G_1, A_1, such that if now p, q be any two numbers of G_1 and $\phi(p)$, $\phi(q)$ the corresponding numbers of A_1, relation (2) still holds. By treating G_1, A_1 as we did G_0, A_0, we obtain two new sequences G_2, A_2; and this process may be continued *ad libitum*. After n steps we have two sequences G_n, A_n, of which the first has a common ratio equal to $\sqrt[2n]{a}$ and the second a common difference b/n. The latter approaches 0 as a limit when n increases indefinitely; whereas the former with increasing n must approach the limit 1. We have then

$$\sqrt[2n]{a} = 1 + \varepsilon_n,$$

where ε_n may be made smaller than any assigned positive number by taking n sufficiently large.

Now let x be any (finite) positive real number whatever. Either x is contained in some G_n, and then $\phi(x)$ is already defined as the corresponding number of A_n; or x is not contained in any G_n no matter how large we take n. In the latter case suppose x lies between the two consecutive numbers x_n and $x_n(1+\varepsilon_n)$ of G_n; then we have

$$x - x_n < x_n \varepsilon_n.$$

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*Cf. e. g. Fischer and Schwatt, *Higher Algebra*, p. 267, or Stolz, *loc. cit.* p. 129.

Since ϵ_n can be made as small as desired by taking n sufficiently large, it follows that as n increases indefinitely the sequence

$$x_0, x_1, x_2, \ldots, x_n$$

approaches x as a limit. From a fundamental theorem on limits it follows that the corresponding sequence

$$\phi(x_0), \phi(x_1), \phi(x_2), \ldots, \phi(x_n)$$

approaches a limit, which we *define* as $\phi(x)$.

The function $\phi(x)$, $=y$ say, is now defined for all values of x such that $0<x<+\infty$. Moreover y, $=\phi(x)$, *is clearly a one-valued and continuous function in this interval, satisfying the functional relation* (2).

Similar reasoning in the opposite direction (*i. e.* from the arithmetical progression to the geometric) shows that *the inverse function* $x=\phi^{-1}(y)=\psi(y)$, *say, is one-valued and continuous in the interval* $-\infty<y<+\infty$ *and satisfies the functional relation* $\psi(\mu)\psi(\nu)=\psi(\mu+\nu)$.

We must now prove that $\phi(x)$ is identical with $\log_k(x)$, or, what is the same, that $\psi(y)$ is identical with k^y, where the base k is $\phi(1)$.[*]

From (2) we obtain readily

$$\phi(x^m)=m\phi(x) \tag{3}$$

when m is any positive integer. From this we may show that *every positive number a has just one mth root.* For, if $\phi(a)=b$, the number $x=\psi(b/m)$ is easily shown to satisfy the relation $x^m=a$, and conversely the only positive number x satisfying this relation is $\psi(b/m)$.[†]

Further, if the *meaning* of a^α, where α is any rational number (positive, negative, or zero), be defined as usual, this proves the *existence* of all rational powers of a.

Now, (3) *holds for all rational values of m.* For, replacing x by $x^{1/t}$ and m by t in (3) we obtain

$$\phi(x^{1/t})=(1/t)\phi(t)$$

and thence easily

$$\phi(x^{s/t})=(s/t)\phi(x),$$

s, t being integers, and s positive, which is no restriction. By replacing x and m by $k=\psi$ (1) and y respectively in (3), we obtain

$$\phi(k^y)=y\phi(k)=y, \text{ whence } k^y=\psi(y)$$

for all rational values of y. If the sequence of rational numbers

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[*] Cf. in this connection, Tannery, *Theorie des functions d'une variable*, p. 120, §82.

[*] Bradshaw, *loc. cit.*, page 58.

$$y_0, y_1, y_2, \ldots, y_n$$

approaches a limit y, when n increases indefinitely, the sequence

$$\psi(y_0), \psi(y_1), \psi(y_2), \ldots, \psi(y_n)$$

approaches the limit $\psi(y)$, and hence *the identical sequence*

$$k^{y_0}, k^{y_1}, k^{y_2}, \ldots k^{y_n}$$

*approaches a limit.** This limit is *defined* as k^y.† Hence for all real values of y

$$\psi(y) = k^y \text{ and hence } \phi(x) = \log_k(x).$$

EVANSTON, ILLINOIS, *September 21, 1903.*

CONVERSE AND OPPOSITE PROPOSITIONS.

By C. M. HIMEL, Baker-Himel School, Knoxville, Tenn.

Wentworth, in the revised edition of his Plane Geometry, page 5, makes the following confusing statements:

"If a direct proposition and its opposite are true, the converse proposition is true; and if a direct proposition and its converse are true, the opposite proposition is true."

"Thus, if it were true that

1. If an animal is a horse, the animal is a quadruped;
2. If an animal is not a horse, the animal is not a quadruped;

it would follow that

3. If an animal is a quadruped, the animal is a horse.

Moreover, if 1 and 3 are true, then 2 would be true."

The statements should read: *Whether a direct proposition is true or not, if the converse is true the opposite is true; if the opposite is true the converse is true.*

Thus, in the above example, if 1 be true, then 3 would follow; if 3 be true, then 2 would be true. Consider the general case:

1. Proposition: If a is b, then c is d.
2. Converse: If c is d, then a is b.
3. Opposite: If a is not b, then c is not d.

We may prove that if either of the last two, irrespective of the first, is true, the other is also true.

First. Suppose you know that the converse is true, and you wish to prove

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*This required proof.

†Cf. Stolz, *loc. cit.*, p. 138; Tannery, *Theorie des fonctions d'une variable*, p. 114.